Linear Algebra II

08/04/2013, Monday, 9:00-12:00

1 Inner product spaces and Gram-Schmidt process $3 + 6 + (6 + 3) = 18$ pts

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(a) Let v_1, v_2, and v_3 be nonzero vectors of an inner product space V. Show that v_1, v_2, and v_3 are linearly independent if $\{v_1, v_2, v_3\}$ is an orthogonal set of vectors.

(b) Consider the vector space P_4. Let

$$\langle p, q \rangle = p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3$$

where $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$ and $q(x) = q_0 + q_1x + q_2x^2 + q_3x^3$. Show that $\langle \cdot, \cdot \rangle$ is an inner product on P_4.

(c) Consider the vector space P_4. Let S be the subspace spanned by the vectors 1, $1 + x$, and $(1 + x)^2$.

(i) By applying the Gram-Schmidt process, find an orthonormal basis for the subspace S.

(ii) Find the vector p in S that is closest to the vector $(1 + x)^3$.

REQUIRED KNOWLEDGE: **inner product, Gram-Schmidt process, least-squares approximation**

SOLUTION:

(1a):

Since $\{v_1, v_2, v_3\}$ is an orthogonal set of vectors, we know that

$$\langle v_i, v_j \rangle = 0 \tag{1}$$

whenever $i \neq j$. Also, we know that

$$\|v_i\|^2 = \langle v_i, v_i \rangle \neq 0 \tag{2}$$

since v_i is nonzero. Let c_1, c_2, and c_3 be scalars such that

$$c_1v_1 + c_2v_2 + c_3v_3 = 0.$$

Then, it follows from (1) that

$$0 = \langle v_i, c_1v_1 + c_2v_2 + c_3v_3 \rangle = c_i\langle v_i, v_i \rangle = c_i\|v_i\|^2$$

for all $i \in \{1, 2, 3\}$. In view of (2), we get $c_i = 0$ for all $i \in \{1, 2, 3\}$. Therefore, the vectors v_1, v_2, and v_3 are linearly independent.

(1b):

An inner product on P_4 satisfies the following properties:

i. $\langle p, p \rangle \geqslant 0$ and $\langle p, p \rangle = 0$ if and only if $p = 0$

ii. $\langle p, q \rangle = \langle q, p \rangle$

iii. $\langle p, \alpha q + \beta r \rangle = \alpha\langle p, q \rangle + \beta\langle p, r \rangle$

for all p, q, $r \in P_4$ and α, $\beta \in \mathbb{R}$.

Note that $\langle p, p\rangle = p_0^2 + p_1^2 + p_2^2 + p_3^2 \geqslant 0$ for $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$. Moreover, $p_0^2 + p_1^2 + p_2^2 + p_3^2 = 0$ if and only if $p_0 = p_1 = p_2 = p_3 = 0$, i.e. $p(x) = 0$. Therefore, the first condition is met.

To prove the second condition is satisfied, let $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$ and $q(x) = q_0 + q_1x + q_2x^2 + q_3x^3$. Note that

$$\begin{aligned}\langle p, q\rangle &= p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3\\ &= q_0p_0 + q_1p_1 + q_2p_2 + q_3p_3\\ &= \langle q, p\rangle.\end{aligned}$$

This means that the second condition is satisfied too.

Finally, let $p(x) = p_0 + p_1x + p_2x^2 + p_3x^3$, $q(x) = q_0 + q_1x + q_2x^2 + q_3x^3$, and $r(x) = r_0 + r_1x + r_2x^2 + r_3x^3$. Note that

$$\begin{aligned}\langle p, \alpha q + \beta r\rangle &= p_0(\alpha q_0 + \beta r_0) + p_1(\alpha q_1 + \beta r_1) + p_2(\alpha q_2 + \beta r_2) + p_3(\alpha q_3 + \beta r_3)\\ &= \alpha(p_0q_0 + p_1q_1 + p_2q_2 + p_3q_3) + \beta(p_0r_0 + p_1r_1 + p_2r_2 + p_3r_3)\\ &= \alpha\langle p, q\rangle + \beta\langle p, r\rangle\end{aligned}$$

for all α, $\beta \in \mathbb{R}$. Consequently, the third condition is satisfied.

Therefore, $\langle\cdot,\cdot\rangle$ is an inner product on P_4.

(1c-i):

To apply the Gram-Schmidt process, we first note that

$$\begin{aligned}\langle 1, 1\rangle &= 1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 = 1\\ \langle 1, x\rangle &= 1\cdot 0 + 0\cdot 1 + 0\cdot 0 + 0\cdot 0 = 0\\ \langle 1, x^2\rangle &= 1\cdot 0 + 0\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0\\ \langle x, x\rangle &= 0\cdot 0 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 = 1\\ \langle x, x^2\rangle &= 0\cdot 0 + 1\cdot 0 + 0\cdot 1 + 0\cdot 0 = 0\\ \langle x^2, x^2\rangle &= 0\cdot 0 + 0\cdot 0 + 0\cdot 0 + 1\cdot 1 = 1.\end{aligned}$$

By applying the Gram-Schmidt process, we obtain:

$$u_1 = \frac{1}{\|1\|}$$

$$u_1 = 1$$

$$u_2 = \frac{1 + x - p_1}{\|1 + x - p_1\|} \qquad \begin{aligned} p_1 &= \langle 1 + x, 1\rangle\cdot 1\\ &= 1\\ 1 + x - p_1 &= x\\ \|1 + x - p_1\|^2 &= \langle x, x\rangle = 1\\ \|1 + x - p_1\| &= 1\end{aligned}$$

$$u_2 = x$$

$$u_3 = \frac{(1+x)^2 - p_2}{\|(1+x)^2 - p_2\|} \qquad\qquad \begin{aligned} p_2 &= \langle (1+x)^2, 1\rangle \cdot 1 + \langle (1+x)^2, x\rangle \cdot x \\ &= 1 + 2x \\ (1+x)^2 - p_2 &= x^2 \\ \|(1+x)^2 - p_2\|^2 &= \langle x^2, x^2\rangle = 1 \\ \|(1+x)^2 - p_2\| &= 1 \end{aligned}$$

$$u_3 = x^2.$$

(1c-ii):

The vector p in S that is closest to the vector $(1+x)^3$ can be found by projection as follows:

$$p = \langle (1+x)^3, 1\rangle \cdot 1 + \langle (1+x)^3, x\rangle \cdot x + \langle (1+x)^3, x^2\rangle \cdot x^2.$$

Note that

$$\langle x^3, 1\rangle = \langle x^3, x\rangle = \langle x^3, x^2\rangle = 0.$$

Therefore, we have

$$p = 1 + 3x + 3x^2.$$

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2 Eigenvalues and diagonalization $4 + 4 + (2 + 8) = 18$ pts

(a) Let $M \in \mathbb{C}^{m\times m}$ be of the form $M = A + iB$ where $A, B \in \mathbb{R}^{m\times m}$. Show that M is Hermitian if and only if $A = A^T$ and $B = -B^T$.

(b) Let $M \in \mathbb{R}^{m\times m}$ be a matrix with $M = \alpha M^T$ for some real number α. Show that M is unitarily diagonalizable.

(c) Consider the matrix

$$M = \begin{bmatrix} 3 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}.$$

(i) Show that the eigenvalues are -1, -1 and 5.

(ii) Find an orthogonal matrix U that diagonalizes M.

REQUIRED KNOWLEDGE: **eigenvalues, eigenvectors, Hermitian matrices, orthogonal matrices, diagonalization by orthogonal matrices.**

SOLUTION:

(2a):

The matrix M is Hermitian if $M = M^H$ where M^H denotes the conjugate transpose. Then, we have

$$\begin{aligned} M = M^H \quad &\iff \quad A + iB = (A + iB)^H \\ &\iff \quad A + iB = A^T - iB^T \\ &\iff \quad A = A^T \text{ and } B = -B^T. \end{aligned}$$

(2b):

A matrix is unitarily diagonalizable if and only if it is normal. Note that

$$MM^H = MM^T = \alpha(M^T)^2 \quad \text{and} \quad M^H M = M^T M = \alpha(M^T)^2.$$

Therefore, M is normal and hence unitarily diagonalizable.

(2c-i):

The characteristic polynomial of M is given by

$$\begin{aligned} p_M(\lambda) = \det(M - \lambda I) &= \det\left(\begin{bmatrix} 3-\lambda & -2 & 2 \\ -2 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{bmatrix} \right) \\ &= \lambda^2(3-\lambda) + 4 + 4 + 4\lambda - (3-\lambda) + 4\lambda \\ &= -\lambda^3 + 3\lambda^2 + 9\lambda + 5. \end{aligned}$$

Note that $p_M(5) = -125 + 3 \cdot 25 + 9 \cdot 5 + 5 = -125 + 75 + 45 + 5 = 0$. Therefore, 5 is an eigenvalue. Note also that

$$\frac{p_M(\lambda)}{5-\lambda} = \lambda^2 + 2\lambda + 1.$$

Then, we can conclude that the two other eigenvalues are -1 and -1.

(2c-ii):

For the eigenvalue 5, we can find an eigenvector by solving the following linear equation:

$$0 = (M - 5I)x = \begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This results in

$$x = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

For the eigenvalue -1, we need to find two orthonormal eigenvectors. To do so, we solve the following linear equations:

$$0 = (M + I)x = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This results in $x_2 = 2x_1 + x_3$ and leads to, for instance, the eigenvectors:

$$y = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad z = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

To orthonormalize these two vectors, we can use the Gram-Schmidt process:

$$u_1 = \frac{y}{\|y\|} \qquad \|y\|^2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = 2$$

$$\|y\| = \sqrt{2}$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$u_2 = \frac{z - p_1}{\|z - p_1\|} \qquad p_1 = \langle z, u_1 \rangle \cdot u_1$$

$$= \frac{1}{2} \left(\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}^T \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$z - p_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

$$\|z - p_1\|^2 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}^T \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 3$$

$$\|z - p_1\| = \frac{1}{\sqrt{3}}$$

$$u_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Therefore, we have

$$\begin{bmatrix} 3 & -2 & 2 \\ -2 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix}^{-1}$$

3 Positive definiteness $6 + 12 = 18$ pts

(a) Let $M \in \mathbb{R}^{m\times m}$ be a symmetric matrix. Show that M^2 is positive definite if and only if M is nonsingular.

(b) Check if the matrix

$$M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

is positive definite or not.

Required Knowledge: **positive/definite matrices, leading principal minor test for positive definiteness.**

Solution:

(3a):

Note that $M^2 = M^T M$ since M is symmetric. Then, we have

$$\begin{aligned} M^2 \text{ is positive definite} &\iff x^T M^2 x > 0 \text{ for all nonzero } x \in \mathbb{R}^m \\ &\iff x^T M^T M x > 0 \text{ for all nonzero } x \in \mathbb{R}^m \\ &\iff \|Mx\|^2 > 0 \text{ for all nonzero } x \in \mathbb{R}^m \\ &\iff Mx \neq 0 \text{ for all nonzero } x \in \mathbb{R}^m \\ &\iff M \text{ is nonsingular} \end{aligned}$$

(3b):

A symmetric matrix M is positive definite if and only if all its leading principal minors are positive. Note that the leading principal minors can be computed as follows:

$$\begin{aligned} \det(1) &= 1 \\ \det\left(\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\right) &= 2 - 1 = 1 \\ \det\left(\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}\right) &= 1\cdot 2\cdot 3 + 1\cdot 2\cdot 1 + 1\cdot 1\cdot 2 - 1\cdot 2\cdot 1 - 1\cdot 2\cdot 2 - 1\cdot 1\cdot 3 \\ &= 6 + 2 + 2 - 2 - 4 - 3 = 1. \end{aligned}$$

Since all leading principal minors are positive, the matrix is positive definite.

4 Singular value decomposition $13 + 5 = 18$ pts

Consider the matrix

$$M = \begin{bmatrix} a & -b & 0 \\ b & a & -b \\ 0 & b & a \end{bmatrix}$$

where a and b are positive real numbers.

(a) Find a singular value decomposition of M.

(b) Find the best rank 2 approximation of M.

Required Knowledge: **singular value decomposition, lower rank approximations.**

Solution:

(4a):

Note that

$$M^T M = \begin{bmatrix} a^2+b^2 & 0 & -b^2 \\ 0 & a^2+2b^2 & 0 \\ -b^2 & 0 & a^2+b^2 \end{bmatrix}.$$

Then, the characteristic polynomial of $M^T M$ can be found as

$$\begin{aligned} p_{M^T M}(\lambda) &= \det\left(\begin{bmatrix} a^2+b^2-\lambda & 0 & -b^2 \\ 0 & a^2+2b^2-\lambda & 0 \\ -b^2 & 0 & a^2+b^2-\lambda \end{bmatrix} \right) \\ &= (a^2+2b^2-\lambda)[(a^2+b^2-\lambda)^2 - b^4] \end{aligned}$$

by expanding the determinant with respect to the second row (or column). Note that

$$\begin{aligned} p_{M^T M}(\lambda) &= (a^2+2b^2-\lambda)[(a^2+b^2-\lambda)^2 - b^4] \\ &= (a^2+2b^2-\lambda)(a^2+b^2-\lambda+b^2)(a^2+b^2-\lambda-b^2). \end{aligned}$$

Since b is not zero, we get

$$\lambda_1 = \lambda_2 = a^2 + 2b^2 > \lambda_3 = a^2.$$

Since a is a positive real number, this results in the following singular values:

$$\sigma_1 = \sigma_2 = \sqrt{a^2+2b^2} > \sigma_3 = a.$$

Next, we need to diagonalize $M^T M$ in order to obtain the orthogonal matrix V. To do so, we first compute eigenvectors of $M^T M$.

For the eigenvalue $a^2 + 2b^2$, we have

$$0 = \left(M^T M - (a^2+2b^2)I\right)x = \begin{bmatrix} -b^2 & 0 & -b^2 \\ 0 & 0 & 0 \\ -b^2 & 0 & -b^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

This results in, for instance, the following orthonormal eigenvectors

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and} \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

since b is not zero.

For the eigenvalue a^2, we have

$$0 = \left(M^T M - a^2 I\right) v = \begin{bmatrix} b^2 & 0 & -b^2 \\ 0 & 2b^2 & 0 \\ -b^2 & 0 & b^2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

This yields the following eigenvector

$$v_3 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

since b is not zero. Hence, we get

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix}.$$

Note that the rank of M is equal to the number of nonzero singular values. Thus, $r = \text{rank}(M) = 3$. By using the formula

$$u_i = \frac{1}{\sigma_i} M v_i$$

for $i = 1, 2, 3$, we obtain

$$u_1 = \frac{1}{\sqrt{2(a^2+2b^2)}} \begin{bmatrix} a & -b & 0 \\ b & a & -b \\ 0 & b & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{2(a^2+2b^2)}} \begin{bmatrix} a \\ 2b \\ -a \end{bmatrix}$$

$$u_2 = \frac{1}{\sqrt{a^2+2b^2}} \begin{bmatrix} a & -b & 0 \\ b & a & -b \\ 0 & b & a \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{a^2+2b^2}} \begin{bmatrix} -b \\ a \\ b \end{bmatrix}$$

$$u_3 = \frac{1}{\sqrt{2}a} \begin{bmatrix} a & -b & 0 \\ b & a & -b \\ 0 & b & a \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Consequently, one singular value decomposition can be given by:

$$\begin{bmatrix} a & -b & 0 \\ b & a & -b \\ 0 & b & a \end{bmatrix} = \begin{bmatrix} \frac{a}{\sqrt{2(a^2+2b^2)}} & \frac{-b}{\sqrt{a^2+2b^2}} & \frac{1}{\sqrt{2}} \\ \frac{2b}{\sqrt{2(a^2+2b^2)}} & \frac{a}{\sqrt{a^2+2b^2}} & 0 \\ \frac{-a}{\sqrt{2(a^2+2b^2)}} & \frac{b}{\sqrt{a^2+2b^2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{a^2+2b^2} & 0 & 0 \\ 0 & \sqrt{a^2+2b^2} & 0 \\ 0 & 0 & a \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

(4b):

The best rank 2 approximation can be obtained as follows:

$$\tilde{M} = \begin{bmatrix} \frac{a}{\sqrt{2(a^2+2b^2)}} & \frac{-b}{\sqrt{a^2+2b^2}} & \frac{1}{\sqrt{2}} \\ \frac{2b}{\sqrt{2(a^2+2b^2)}} & \frac{a}{\sqrt{a^2+2b^2}} & 0 \\ \frac{-a}{\sqrt{2(a^2+2b^2)}} & \frac{b}{\sqrt{a^2+2b^2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{a^2+2b^2} & 0 & 0 \\ 0 & \sqrt{a^2+2b^2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{a}{\sqrt{2}} & -b & 0 \\ \frac{2b}{\sqrt{2}} & a & 0 \\ \frac{-a}{\sqrt{2}} & b & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{-1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{a}{2} & -b & \frac{a}{2} \\ b & a & b \\ \frac{-a}{2} & b & \frac{-a}{2} \end{bmatrix}.$$

5 Cayley-Hamilton theorem and Jordan canonical form $6 + 12 = 18$ pts

Consider the matrix

$$M = \begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix}$$

where a and b are real numbers with $a \neq 0$.

(a) By using Cayley-Hamilton theorem, show that $(M^2 - I)^{999} = 0$.

(b) Put the matrix M into Jordan canonical form.

REQUIRED KNOWLEDGE: **Cayley-Hamilton theorem, Jordan canonical form.**

SOLUTION:

(5a):

The characteristic polynomial of M can be found as:

$$p_M(\lambda) = \det(M - \lambda I) = \det\left(\begin{bmatrix} 1-\lambda & a & b \\ 0 & 1-\lambda & a \\ 0 & 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^3.$$

Note that $M^2 - I = (M-I)(M+I) = (M+I)(M-I)$. Thus, we get $(M^2 - I)^{999} = (M-I)^{999}(M+I)^{999}$. It follows from Cayley-Hamilton theorem that $(M-I)^3 = 0$. Therefore, we get $(M^2 - I)^{999} = 0$.

(5b):

Since $p_M(\lambda) = (1-\lambda)^3$, there is only one eigenvalue $\lambda = 1$ with multiplicity 3. To find the linearly independent eigenvectors, we need to solve the following linear equation:

$$0 = (M-I)x = \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Since a is not zero, this yields only one linearly independent eigenvector:

$$x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

This means that the Jordan form should be $\left[\begin{smallmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{smallmatrix}\right]$. Note that

$$(M-I)^2 = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad (M-I)^3 = 0.$$

Then, we need to solve $(M-I)^2 v = x$, that is

$$\begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Since a is not zero, one solution can be found as:

$$v = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a^2} \end{bmatrix}.$$

Note that

$$(M-I)v = \begin{bmatrix} 0 & a & b \\ 0 & 0 & a \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a^2} \end{bmatrix} = \begin{bmatrix} \frac{b}{a^2} \\ \frac{1}{a} \\ 0 \end{bmatrix} \quad \text{and} \quad (M-I)^2 v = \begin{bmatrix} 0 & 0 & a^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \frac{1}{a^2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the Jordan canonical form can be given as:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & \frac{b}{a^2} & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{a^2} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \frac{b}{a^2} & 0 \\ 0 & \frac{1}{a} & 0 \\ 0 & 0 & \frac{1}{a^2} \end{bmatrix}^{-1}.$$